Symmetric Matrices

Defn: A matrix M is symmetric when $M^T = M$.

NB: Because the transpose of an $m \times n$ matrix is $n \times m$, the symmetry condition $M^T = M$ implies M is square.

Ex:
$$\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{is symmetric.}$$

$$\underbrace{\begin{bmatrix} 3 & -5 \\ -1 & 0 \end{bmatrix}}^T = \underbrace{\begin{bmatrix} 3 & -1 \\ -5 & 0 \end{bmatrix}} \quad \text{is } \underline{\text{NOT}} \text{ symmetric.}$$

M^T is obtained from M by swapping roles of rows and columns.

$$\boxed{[m_{ij}]^T \underset{*}{=} [m_{ji}]}$$

Ex: The 2×2 real symmetric matrices are:

$$\mathrm{Symm}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

Note:
$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} a+x & \boxed{b+y} \\ \boxed{b+y} & c+z \end{bmatrix}$$

$$k\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} ka & \boxed{kb} \\ \boxed{kb} & kc \end{bmatrix}, \quad \text{so } \mathrm{Symm}_2(\mathbb{R}) \leq M_{2\times 2}(\mathbb{R}).$$

Prop: Suppose A, B are $m \times n$ matrices and k is a scalar.
$$(A + kB)^T = A^T + kB^T.$$

pf:
$$\begin{aligned}
(A+kB)^T &= \left([a_{ij}] + k[b_{ij}]\right)^T \\
&= [a_{ij} + kb_{ij}]^T \\
&= [a_{ji} + kb_{ji}] \\
&= [a_{ji}] + k[b_{ji}] \\
&= [a_{ij}]^T + k[b_{ij}]^T = A^T + kB^T \qquad \blacksquare
\end{aligned}$$

Cor: If A, B are symmetric and k is a scalar, then $A + kB$ is symmetric.

Pf: $(A + kB)^T = A^T + kB^T = A + kB$ $\square$

Cor: The set of symmetric matrices is a subspace of the space of square matrices for every n.

$$\left(\text{i.e. } Symm_n(\mathbb{R}) \leq M_{n\times n}(\mathbb{R})\right).$$

Q: What is a nice basis of $Symm_n(\mathbb{R})$ (or $Symm_n(\mathbb{C})$)?

A: For $n=2$: $\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

so $\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{M_{1,1}}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{M_{1,2}}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{M_{2,2}} \right\}$ span $Symm_n(\mathbb{F})$.

Lin. ind. follows because $kM_{i,j}$ has zeroes everywhere except (i,j) and (j,i) entries...

So $E_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis.

For $n=3$:

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = a\overset{M_{1,1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} + b\overset{M_{1,2}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} + c\overset{M_{1,3}}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$
$$+ d\underset{M_{2,2}}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}} + e\underset{M_{2,3}}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} + f\underset{M_{3,3}}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$E_3 = \{ M_{i,j} : 1 \leq i \leq j \leq 3 \}$ is a basis of $Symm_3(\mathbb{F})$.

In general: $E_n = \{ M_{i,j} : 1 \leq \boxed{i \leq j} \leq n \}$ is a basis of $Symm_n(\mathbb{F})$

where $M_{i,j}$ has 1's in (i,j) and (j,i), and 0's everywhere else. $\square$

Cor: $\dim(Symm_n(\mathbb{R})) = \dfrac{n(n-1)}{2} + n = \dfrac{n(n+1)}{2}$.

Q: Is the product of symmetric matrices also symmetric?

Prop: Suppose A is an $(m \times k)$-matrix and B is a $(k \times n)$ matrix

Then $(AB)^T = B^T A^T$.

Pf: On -hold. $\boxed{\text{☺}}$

$(AB)^{-1} = B^{-1}A^{-1}$

Special Case: if $m = k = n = 2$:

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)^T = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix}^T$$

$$= \begin{bmatrix} \boxed{ax + bz} & \boxed{cx + dz} \\ \boxed{ay + bw} & cy + dw \end{bmatrix}$$

$\| $

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} x & z \\ y & w \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{xa + zb} & \boxed{xc + zd} \\ \boxed{ya + wb} & \boxed{yc + wd} \end{bmatrix}$$

So if A and B are symmetric,

$$(AB)^T = B^T A^T = BA \cancel{= AB}$$

Not always true ☹.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ Both ARE symmetric.

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$ NOT symmetric. $\boxed{\text{☺}}$

$\boxed{(AB)^T = B^T A^T}$ ✱ ← always true

Prop: If A is invertible, then $(A^{-1})^T = (A^T)^{-1}$

Pf: $\underbrace{(A^{-1})^T}_{B} \underbrace{A^T} = (AA^{-1})^T = I^T = I$ $\quad \therefore (A^T)^{-1} = (A^{-1})^T$ $\boxed{\text{☺}}$

Bad News: Products of symmetric matrices aren't symmetric.

Good News: We can still build symmetric matrices via product...

Consider any square matrix A.

$$(A^TA)^T = A^T(A^T)^T = A^TA$$

So A^TA is always symmetric.

Ex: $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$. $A^TA = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$

Q: What can the eigenvalues of a symmetric matrix be?

A (Forthcoming): If A is a real symmetric matrix, then the eigenvalues of A are all real!

↳ to give the full answer, we need to study more about the complex vector spaces...

Defn: Let $z = a+bi$ be a complex number (w/ $a, b \in \mathbb{R}$).
The complex conjugate of z is $\bar{z} := \overline{(a+bi)} = a - bi$.

Ex: $\overline{3-i} = 3+i$, $\overline{5+7i} = 5-7i$, $\overline{\pi i} = -\pi i$, $\bar{e} = e$

Lem: $\bar{z} = z$ if and only if $z \in \mathbb{R}$.

Pf: ($\Rightarrow$): If $\overline{a+bi} = a+bi$, then $a - bi = \overline{a+bi} = a+bi$,
so $2bi = 0$ yields $b = 0$.

($\Leftarrow$): $\bar{a} = \overline{a+0i} = a - 0i = a$

NB: If $A \in M_{m\times n}(\mathbb{C})$, we can write $A = \mathrm{Re}(A) + i\,\mathrm{Im}(A)$
where both $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ are real matrices.

Ex: $A = \begin{bmatrix} 1+i & 1-i \\ 3+2i & 5-i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} i & -i \\ 2i & -i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} + i \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$.

$$\mathrm{Re}(A) = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}, \quad \mathrm{Im}(A) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}.$$

Point: We can extend the definition of conjugate to matrices!

$$\overline{A} = \overline{\mathrm{Re}(A) + i\,\mathrm{Im}(A)} = \mathrm{Re}(A) - i\,\mathrm{Im}(A)$$